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LETTER TO THE EDITOR

The quantum group $T_h GL(n)$

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Abstract. Let G be a Lie group and G_q be its quantum analogue. In general, it is not known how to construct a quantum analogue of the Lie group TG, the tangent bundle of G. For the case $G = GL(n)$, the quantum group $T_h GL(n)$, $h = \ln q$, is described below.

Let G be a Lie group and G_q be its quantum analogue (Drinfel'd 1986). Let TG denote the tangent bundle of G; it is a Lie group with the multiplication

$$\begin{pmatrix} g_1 \\ v_1 \end{pmatrix}\begin{pmatrix} g_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} g_1 g_2 \\ L_{g_1}(v_2) + R_{g_2}(v_1) \end{pmatrix} \qquad g_i \in G \qquad v_i \in T_{g_i} G \tag{1}$$

where L_g and R_g stand for the left and right translations by $g \in G$, respectively. Given G_q, how to *derive* a quantum group version of TG? It is not known, although one may expect that the answer amounts to some simple operation on R-matrices. This expectation is based on what happens in the quasiclassical limit of quantum group structures, i.e. multiplicative Poisson brackets (PB) on Lie groups. Recall that a PB on G is called multiplicative (equivalently, G is called a Poisson–Lie group) if the multiplication map

$$\mu : G \times G \to G \tag{2}$$

is a Poisson map, with the Poisson structure on $G \times G$ being that of the direct product. Now, if P is a Poisson manifold then TP can also be made into a Poisson manifold (Kupershmidt 1986), with the PB $\{,\}_{TP}$ on TP being defined by the formulae

$$\begin{aligned} &\{F_1, F_2\}_{TP} = 0 \qquad \{F_1, d(F_2)\}_{TP} = \{F_1, F_2\}_P \\ &\{d(F_1), d(F_2)\}_{TP} = d(\{F_1, F_2\}_P) \qquad \forall F_1, F_2 \in C^\infty(P). \end{aligned} \tag{3}$$

Moreover, since formulae (3) are natural, i.e. compatible with Poisson maps, it follows that when P is a Lie group G and the PB $\{,\}_G$ is multiplicative then so is $\{,\}_{TG}$.

For the case $G = GL(n)$, the quasiclassical formulae (3) *can* be quantized. This is how.

Recall the commutation relations for a quantum matrix $m \in GL_q(n)$ (Drinfel'd 1986). In the notation of (Kupershmidt 1990), let $\Lambda = (\lambda(i,j))$ be a skewsymmetric matrix, and let $A_q^{n|0}$ and $A_q^{0|n}$ be the polynomial rings on two quantum n-dimensional spaces given by the generators and relations (Manin 1988)

$$x_i x_j - q^{\lambda(i,j)} x_j x_i = 0 : A_q^{n|0} \tag{4.1}$$

$$\xi_i \xi_j + q^{\lambda(j,i)} \xi_j \xi_i = 0 : A_q^{0|n} \qquad 1 \leq i, j \leq n. \tag{4.2}$$

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Demanding that the components of the vectors

$$x' = mx \qquad \xi' = m\xi \tag{5}$$

again satisfy the commutation relations (4), one arrives at the commutation relations for m:

$$m_{i\alpha}m_{j\beta}[q^{\lambda(\alpha,\beta)}+q^{\lambda(\beta,\alpha)}] = m_{j\beta}m_{i\alpha}[q^{\lambda(i,j)}+q^{\lambda(j,i)}]+m_{j\alpha}m_{i\beta}[q^{\lambda(i,j)+\lambda(\alpha,\beta)}-q^{\lambda(j,i)+\lambda(\beta,\alpha)}]. \tag{6}$$

In the quasiclassical limit $q = e^h = 1 + h + O(h^2)$, formulae (4), (6) become the PB formulae

$$\{x_i, x_j\} = \lambda(i,j)x_ix_j \tag{4.1'}$$

$$\{\xi_i, \xi_j\} = \lambda(j,i)\xi_i\xi_j \tag{4.2'}$$

$$\{m_{i\alpha}, m_{j\beta}\} = [\lambda(i,j)+\lambda(\alpha,\beta)]m_{i\beta}m_{j\alpha}. \tag{6'}$$

The standard choice

$$\lambda(i,j) = \text{constant} \times \text{sgn}(i-j) \tag{7}$$

results if one requires the PB (6′) to satisfy the Jacobi identities.

We assume that the Manin receipe (5) produces also the desired quantization of TGL(n). Thus, we need first to find $TA_q^{n|0}$ and $TA_q^{0|n}$. To do that we apply the T-operation (3) to formulae (4′), (6′), resulting in

$$\{x_i, x_j\} = 0, \{x_i, y_j\} = \lambda(i,j)x_ix_j \qquad \{y_i, y_j\} = \lambda(i,j)(x_iy_j + y_ix_j) \tag{8.1}$$

$$\{\xi_i, \xi_j\} = 0, \{\xi_i, \eta_j\} = \lambda(j,i)\xi_i\xi_j \qquad \{\eta_i, \eta_j\} = \lambda(j,i)(\xi_i\eta_j + \eta_i\xi_j) \tag{8.2}$$

$$\begin{gathered}\{m_{i\alpha}, m_{j\beta}\} = 0 \qquad \{m_{i\alpha}, M_{j\beta}\} = [\lambda(i,j)+\lambda(\alpha,\beta)]m_{i\beta}m_{j\alpha} \\ \{M_{i\alpha}, M_{j\beta}\} = [\lambda(i,j)+\lambda(\alpha,\beta)](m_{i\beta}M_{j\alpha} + M_{i\beta}m_{j\alpha}).\end{gathered} \tag{9}$$

As a quantum analogue of formulae (8) we set

$$\begin{gathered}[x_i, x_j] = 0 \qquad [x_i, y_j] = h\lambda(i,j)x_ix_j \\ [y_i, y_j] = h\lambda(i,j)(x_iy_j + x_jy_i + Ehx_ix_j)\end{gathered} \tag{10.1}$$

$$\begin{gathered}\xi_i\xi_j + \xi_j\xi_i = 0 \qquad \xi_i\eta_j + \eta_j\xi_i = h\lambda(j,i)\xi_i\xi_j \\ \eta_i\eta_j + \eta_j\eta_i = h\lambda(j,i)(\xi_i\eta_j - \xi_j\eta_i + Eh\xi_i\xi_j)\end{gathered} \tag{10.2}$$

where $h = \ln q$, and E is an arbitrary constant (or a function of h). Formulae (5) become

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} mx \\ Mx + my \end{pmatrix} \qquad \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} m\xi \\ M\xi + m\eta \end{pmatrix}. \tag{11}$$

Thus, we use the matrix representation

$$(m, M) \rightarrow \begin{pmatrix} m & 0 \\ M & m \end{pmatrix} \tag{12}$$

of TGL(n) in GL($2n$). This implies that if the relations (10) were not right, the resulting overdetermined system of relations on m, M would have no solutions. As it is, demanding that the components of the vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{\xi}'$, $\boldsymbol{\eta}'$ (11) satisfy again the relations (10), one obtains the following formulae

$$[m_{i\alpha}, m_{j\beta}] = 0 \qquad [m_{i\alpha}, M_{j\beta}] = h[\lambda(i,j) + \lambda(\alpha,\beta)]m_{j\alpha}m_{i\beta}$$
$$[M_{i\alpha}, M_{j\beta}] = h[\lambda(i,j) + \lambda(\alpha,\beta)](m_{i\beta}M_{j\alpha} + m_{j\alpha}M_{i\beta} + Ehm_{i\beta}m_{j\alpha}). \tag{13}$$

Obviously, in the quasiclassical limit $h \to 0$, formulae (13) become formulae (9).

Formulae (13) describe the desired quantum group $\mathrm{T}_h\mathrm{GL}(n)$. By construction, as in the case of Manin's interpretation of $\mathrm{GL}_q(n)$, formulae (13) are multiplicative. Also, it is not difficult to show that $\det(m)$ is a central element. Finally, using the diamond lemma (Bergman 1978), one can show that the condition (7) $\lambda(i,j) = \mathrm{constant} \times \mathrm{sgn}(i-j)$ is necessary and sufficient for the PBW property to hold for $\mathrm{T}_h\mathrm{GL}(n)$, with *arbitrary E*.

I conclude with a few remarks.

(A) In addition to $\det(m)$, the element

$$T\det(m) = \sum \frac{\partial \det(m)}{\partial m_{i\alpha}} M_{i\alpha} = \sum M_{i\alpha} \frac{\partial \det(m)}{\partial m_{i\alpha}} \tag{14}$$

is also central. It is likely that $\det(m)$ and $\mathrm{T}\det(m)$ are the generators of the polynomial centre of $\mathrm{T}_h\mathrm{GL}(n)$.

(B) The manifold TP is the fibre over $\mathbf{R}$ of the 1-jet bundle $\pi^1 : J^1\pi \to \mathbf{R}$, of the bundle $\pi : P \times \mathbf{R} \to \mathbf{R}$. When P is a Poisson manifold, the fibres of all the jet bundles $\pi^k : J^k\pi \to \mathbf{R}$ are also naturally Poisson. When $\mathrm{P} = \mathrm{G} = \mathrm{GL}(n)$, it is natural to expect that the higher prolongations $\mathrm{G}^{(k)}$ of G can be quantized for all k, and not only for $k = 1$.

(C) If $\mathscr{G}$ is a Lie algebra of the Lie group G and $r : \mathscr{G}^* \to \mathscr{G}$, is an invertible operator then (Drinfel'd 1985) r is a classical r-matrix iff the bilinear form $(\cdot, \cdot\cdot) = [r^{-1}(\cdot)](\cdot\cdot)$ is a two-cocycle on $\mathscr{G}$. Applying the first prolongation T to $\mathscr{G}$ we obtain (Kupershmidt 1986) the semidirect sum Lie algebra (of the Lie group TG) $\mathscr{G}^{(1)} = \mathscr{G} \ltimes \mathscr{G}^{ab}$ with the two-cocycle that corresponds to the r-matrix

$$\mathrm{T}r = r^{(1)} = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}.$$

Here $\mathscr{G}^{ab}$ is the vector space of $\mathscr{G}$ on which $\mathscr{G}$ acts by the adjoint representation. This suggests that the full quantum R-matrix $\mathrm{T}R = R^{(1)}$ on $\mathrm{U}\mathscr{G} \ltimes \mathrm{U}\mathscr{G}^{ab}$ (or TG) is given by the formula

$$R^{(1)} = \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix}.$$

But this would be in contradiction to the appearance of E in formulae (13).

(D) From formula (12),

$$(m, M)^{-1} = (\bar{m}, \bar{M}) = (m^{-1}, -m^{-1}Mm^{-1}). \tag{15}$$

One can show that the matrix elements of $\bar{m}$ and $\bar{M}$ satisfy the commutation relations (13) with the parameters $\bar{h} = -h$, $\bar{E} = -E$. In the quasiclassical limit (9), the map (15) is, as expected, anti-Poisson.

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